（装订线内不要答题）

# 复旦大学技术科学类

## 2022-2023 学年第一学期《数学分析 B（I)》

## 一元微分学阶段性考试 试卷 参考解答

第 1-12 页（直接在卷面上解答）

课程代码：MATH120016.10-11　　考试形式：☐开卷　☑闭卷　　　　2022 年 11 月 13 日

（本次考试计划时间 180 分钟）

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| 1-1 | 1-2 | 1-3 | 1-4 | 1-5 | 1-6 | 1-7 | 1-9 | | |
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| | | | | | | | | | |
| 2-1-1 | 2-1-2 | 2-2-1 | 2-2-2 | 2-3-1 | 2-3-2 | 2-4 | 2-5 | 2-6-1 | 2-6-2 |
| | | | | | | | | | |
| 3-1-1 | 3-1-2 | 3-2 | | | | | | | 总分 |
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注：各部分题目的每一题与小题都按满分 10 分计算，然后折合成总分 100 分。

### 第一部分 概述与方法说明

（1）证明：（Fermat 引理）函数在可微的极值点必定导数为零。

Fermat引理 $\frac{df}{dx}(x_+)\triangleq\lim\limits_{x\to x_*}\frac{f(x)-f(x_*)}{x-x_*}$, 则有 $\begin{cases}f'_+(x_*)\geqslant 0\\ f'_-(x_*)\leqslant 0\end{cases}$;

$\exists f'(x_*)\in\mathbb{R} \Leftrightarrow \exists f'_+(x_*)=f'_-(x_*)\in\mathbb{R}$, 故有 $f'(x_*)=0$

注：

1. Rolle型结论

$\begin{cases}f(x)\in C[a,b]\\ \exists f'(x)\in\mathbb{R},\forall x\in(a,b)\end{cases}$ 则有:如果$(a,b)$中有最值点,则该点导数为零.

2. Cauthy中值定理

$$\frac{\psi(b)-\psi(a)}{\varphi(b)-\varphi(a)}=\frac{\psi'(\xi)}{\varphi'(\xi)},\xi\in(a,b),$$

条件: $\psi(x),\varphi(x)\in C[a,b]$, 内部可导且$\sqrt{\psi'^2(x)+\varphi'^2(x)}>0,\ \forall x\in(a,b)$

（2）证明：（Darboux 定理）导函数具有介值性

$(i)$ 设有$\varphi_+'(a)\cdot\varphi_-'(b)<0,\ \phi(x)$在$(a,b)$上可导，则有 $\exists x_*\in(a,b),\ s.t.\ \varphi'(x_*)=0$

不妨设$\begin{cases}\varphi_+'(a)<0\\ \varphi_-'(b)>0\end{cases}\Rightarrow\ \exists\delta_1,\delta_2,s.t.\begin{cases}\varphi(x)<\varphi(a),\forall x\in(a,a+\delta_1)\\ \varphi(x)<\varphi(b),\forall x\in(b-\delta_2,b)\end{cases}$

$\Rightarrow\ \varphi(x)$在$x_*\in(a,b)$处取最小值，由于$\varphi$可导，由$Fermat$引理，$\varphi'(x_*)=0$

$(ii)\ \varphi(x)=\gamma x-f(x),\ \forall\gamma\in\left(f_+'(a),f_-'(b)\right)$

$s.t.\ \begin{cases}\varphi_+'(a)=f_+'(a)-\gamma<0\\ \varphi_-'(b)=f_-'(b)-\gamma>0\end{cases}$

则有 $\exists x_\gamma\in(a,b),\ s.t. f'(x_\gamma)=\gamma$

（3）阐述：连续函数在定义域内部具有最值点的三种充分性依据

$(i)$ $Rolle$定理 $f(a)=f(b)$

$(ii)$ Darboux定理 $f_+'(a)\cdot f_-'(b)<0$

$(iii)$ 能量衰减机制 $\begin{cases}\varphi(a)=0\\ \dfrac{d\varphi^2}{dx}(b)<0\end{cases}$

注：由极限保号性：$\lim\limits_{x\to b-0}\dfrac{\varphi^2(x)-\varphi^2(b)}{x-b}<0$

（4）证明：对于任意的基本/Cauchy 序列，如果有子列收敛，则此基本序列收敛

$\left.\begin{array}{c}\text{设有}\{x_n\}\text{基本/Cauchy}\\ \exists x_{n_k}\to a\in\mathbb{R},\ as\ k\to+\infty\end{array}\right\}\Rightarrow x_n\to a\in\mathbb{R},\ as\ n\to+\infty$

处理一

$0\leqslant|x_n-a|\leqslant|x_n-x_{n_k}|+|x_{n_k}-a|<\varepsilon+\varepsilon,\ \begin{cases}\forall k>\max\{\tilde{N}_\varepsilon,N_\varepsilon\}\\ \forall n>N_\varepsilon\end{cases}$

处理二

$\left(\forall m,n_{k_\varepsilon}>N_\varepsilon\right)\ \varepsilon>|x_m-x_{n_k}|\to|x_m-a|\le\varepsilon(\forall n\ge N_\varepsilon),\ \text{as}\ k\to+\infty$

（装订线内不要答题）

（5）证明：无界区间上一致连续的函数满足线性增长控制

需证明: $f(x)$在$[a,+\infty)$上一致连续，则有 $|f(x)| \le |f(a)| + K(x-a), \exists K \in \mathbb{R}^+$

$$x \sim \begin{cases} < a+(n_x+1)\delta_\varepsilon \\ \geqslant a+n_x\delta_\varepsilon \end{cases}，有\ x = a+n_x\cdot\delta_\varepsilon+\delta_x,\ \delta_x \in [0,\delta_\varepsilon)$$

$$|f(x)| = \left|f\left(a+n_x\delta_\varepsilon+\delta_x\right)\right| < \left|f\left(a+n_x\delta_\varepsilon\right)\right| + \varepsilon$$

$$< \left|f\left(a+(n_x-1)\delta_\varepsilon\right)\right| + 2\varepsilon < \cdots < |f(a)| + (n_x+1)\varepsilon$$

$$\le |f(a)| + \left(\frac{x-a}{\delta_\varepsilon}+1\right)\cdot\varepsilon \leqslant |f(a)| + \frac{2\varepsilon}{\delta_\varepsilon}(x-a),\ as\ x \gg a$$

（6）设有$f(x)$在$[a,+\infty)$上一致连续，满足$\exists \lim\limits_{n\to+\infty} f(x+n) = A \in \mathbb{R}$， $\forall x \in [a,a+1]$，

证明：$\exists \lim\limits_{x\to+\infty} f(x) = A$

按一致连续，$\forall \varepsilon > 0,\ \exists \delta_\varepsilon > 0,\ s.t.\ |\hat{x}-\tilde{x}| < \delta_\varepsilon,\ |f(\hat{x}) - f(\tilde{x})| < \varepsilon$

可对$[0,1]$做$n_\varepsilon$等分, 分点为$\{x_k\}_{k=0}^{n_\varepsilon}$, 使得子区间宽度小于$\delta_\varepsilon$.
考虑 $\exists \lim\limits_{n\to+\infty} f(x_k+n) = A \in \mathbb{R},\ k = 0,\cdots,n_\varepsilon-1,$
则有 $|f(x_k+n)| < \varepsilon,\ \forall n > \max\left\{N_{\varepsilon,1},\cdots,N_{\varepsilon,n_\varepsilon-1}\right\}$

结合$f(x)$在$[a,+\infty)$上一致连续，则有
$|f(x)-A| \le |f(x) - f(x_k+n)| + |f(x_k+n) - A| < 2\varepsilon,\ \forall x > a + \max\left\{N_{\varepsilon,1},\cdots,N_{\varepsilon,n_\varepsilon-1}\right\},$
式中$|x-(x_k+n)| \le \delta_\varepsilon$

（7）考虑迭代序列 $x_{n+1}=\lambda x_n+\delta_n$ 有界，设有 $\begin{cases}|\lambda|<1\\ \{\delta_n\}_{n\in\mathbb{N}}\text{有界}\end{cases}$，证明：$\{x_n\}_{n\in\mathbb{N}}$ 有界

$$
\begin{aligned}
x_{n+1}&=\lambda x_n+\delta_n\\
&=\lambda\left(\lambda x_{n-1}+\delta_{n-1}\right)+\delta_n=\lambda^2 x_{n-1}+\lambda\delta_{n-1}+\delta_n\\
&=\lambda^2\left(\lambda x_{n-2}+\delta_{n-2}\right)+\lambda\delta_{n-1}+\delta_n\\
&=\lambda^3 x_{n-2}+\lambda^2\delta_{n-2}+\lambda\delta_{n-1}+\delta_n=\cdots=\lambda^n x_1+\left\{\lambda^{n-1}\delta_1+\lambda^{n-2}\delta_2+\cdots+\lambda\cdot\delta_{n-1}+\delta_n\right\}
\end{aligned}
$$

$$RHS^1=\lambda^n x_1;\ RHS^2=\lambda^{n-1}\delta_1+\lambda^{n-2}\delta_2+\cdots+\lambda\cdot\delta_{n-1}+\delta_n$$

$$\left|RHS^1\right|\le|\lambda|^n\cdot|x_1|\le|x_1|;\ \left|RHS^2\right|<\sup_n|\delta_n|\cdot\frac{1}{1-|\lambda|}$$

故，数列 $\{x_n\}$ 有界

（8）设数列 $\{x_n\}_{n\in\mathbb{N}}$ 有界，证明：$\exists\lim\limits_{k\to+\infty}x_{n_k}=\lim\limits_{n\to\infty}\inf\limits_{k\ge n}x_k\in\mathbb{R}$

利用夹逼性，构造

$$\exists x_{m_l}\in\left(\underline{\lim}x_n-\frac{1}{l},\overline{\lim}x_n+\frac{1}{l}\right),\ \forall l\in\mathbb{N}$$

注：

$$(1)\ \begin{cases}\exists x_{n_k}\to\underline{\lim}x_n,\ as\ k\to+\infty\\ \exists x_{m_k}\to\overline{\lim}x_n,\ as\ k\to+\infty\end{cases}$$

(2) 设有 $x_{p_k}\to y_*,\ as\ k\to+\infty$，则有 $\underline{\lim}x_n\le y_*\le\overline{\lim}x_n$

$$\lim_{k\to+\infty}\inf_{m\ge p_k}x_m\le x_{p_k}\le\lim_{k\to+\infty}\sup_{m\ge p_k}x_m,\forall k\in\mathbb{N}\ \Rightarrow\ \underline{\lim}x_n\le y_*\le\overline{\lim}x_n\ (\text{利用保号性})$$

$$(3)\ \exists\lim x_n\in\mathbb{R}\ \Leftrightarrow\ \underline{\lim}x_n=\overline{\lim}x_n\in\mathbb{R}$$

（装订线内不要答题）

## 第二部分 计算与计算证明

### 1. 计算数列极限

（1） $\lim\limits_{n\to+\infty} n\left[\left(1+\frac{1}{n+1}\right)^n - \mathrm{e}\right]$

（2） $\lim\limits_{n\to+\infty}\left(1+\sin\left(\pi\sqrt{1+4n^2}\right)\right)^n$

(1) 分析：考虑 $\left(1+\frac{1}{n+1}\right)^n = e^{n\ln\left(1+\frac{1}{n+1}\right)}$

其中

$$n\ln\left(1+\frac{1}{n+1}\right)=n\left[\frac{1}{n+1}-\frac{1}{2}\left(\frac{1}{n+1}\right)^2+o\left(\frac{1}{n^2}\right)\right]=n\left[\frac{1}{n}\left(1+\frac{1}{n}\right)^{-1}-\frac{1}{2}\frac{1}{n^2}\left(1+\frac{1}{n}\right)^{-2}+o\left(\frac{1}{n^2}\right)\right]$$

$$=n\left[\frac{1}{n}\left(1-\frac{1}{n}+0\left(\frac{1}{n}\right)\right)-\frac{1}{2n^2}+0\left(\frac{1}{n^2}\right)\right]=\left[\frac{1}{n}-\frac{3}{2}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right]=1-\frac{3}{2}\cdot\frac{1}{n}+o\left(\frac{1}{n}\right)$$

$$\Rightarrow\ e^{n\ln\left(1+\frac{1}{n+1}\right)}=e^{1-\frac{3}{2}\frac{1}{n}+o\left(\frac{1}{n}\right)}=e\left(1-\frac{3}{2}\frac{1}{n}+o\left(\frac{1}{n}\right)\right)$$，故有极限为 $-\frac{3}{2}e.$

(2) 分析：原式=$e^{n\ln\left(1+\sin\left(\pi\sqrt{1+4n^2}\right)\right)}$

其中 $\ln\left(1+\sin\left(\pi\sqrt{1+4n^2}\right)\right)=\ln\left[1+\sin\left(2n\pi\left(1+\frac{1}{4n^2}\right)^{\frac{1}{2}}\right)\right]$

$$=\ln\left[1+\sin\left(2n\pi+2n\pi\frac{1}{8n^2}+0\left(\frac{1}{n^2}\right)\right)\right]=\ln\left[1+\frac{\pi}{4n}+0\left(\frac{1}{n^2}\right)\right]=\frac{\pi}{4n}+o\left(\frac{1}{n}\right)$$

$$\Rightarrow e^{n\ln\left(1+\sin\left(\pi\sqrt{1+4n^2}\right)\right)}=e^{n\left(\frac{\pi}{4n}+o\left(\frac{1}{n}\right)\right)}=e^{\frac{\pi}{4}}$$

2. 计算函数极限

（1） $\lim\limits_{x\to 0+0}\left(1+\sqrt{x}\cdot e^{-\frac{1}{x^3}}\cdot\sin\frac{1}{x^4}\right)^{e^{\frac{1}{x^3}}}$

（2） $\lim\limits_{x\to 0}\frac{1-\sqrt{1-x^2}\cos x}{1+x^2-\cos^2 x}$

(1) 分析：原式 $=e^{e^{\frac{1}{x^3}}\ln\left(1+\sqrt{x}\cdot e^{-\frac{1}{x^3}}\cdot\sin\frac{1}{x^4}\right)}$

其中 $e^{\frac{1}{x^3}}\ln\left(1+\sqrt{x}\cdot e^{-\frac{1}{x^3}}\cdot\sin\frac{1}{x^4}\right)=e^{\frac{1}{x^3}}\cdot\sqrt{x}\cdot e^{-\frac{1}{x^3}}\sin\frac{1}{x^4}\left(1+o(1)\right)$

$\Rightarrow e^{e^{\frac{1}{x^3}}\ln\left(1+\sqrt{x}\cdot e^{-\frac{1}{x^3}}\cdot\sin\frac{1}{x^4}\right)}=e^{\sqrt{x}\cdot\sin\frac{1}{x^4}\left(1+o(1)\right)}\to e^0=1$

注： $\sin\frac{1}{x_n^4}=0,\ x_n^4=\frac{1}{n\pi},\ \forall n\in\mathbb{N}$

(2) 分析：

$1+x^2-\cos^2 x=1+x^2-\left(1-\frac{1}{2}x^2+o\left(x^3\right)\right)^2=1+x^2-\left[1-x^2+o\left(x^3\right)\right]=2x^2+o\left(x^3\right).$

$1-\sqrt{1-x^2}\cos x=1-\left(1-x^2\right)^{\frac{1}{2}}\left(1-\frac{1}{2}x^2+o\left(x^3\right)\right)=1-\left(1-\frac{1}{2}x^2+o\left(x^3\right)\right)\left(1-\frac{1}{2}x^2+o\left(x^3\right)\right)$

$=1-\left[1-x^2+o\left(x^3\right)\right]=x^2+o\left(x^3\right)$

$\Rightarrow$ 原式 $=\frac{x^2+o\left(x^3\right)}{2x^2+o\left(x^3\right)}\to\frac{1}{2},\ as\ x\to 0.$

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3. 无限小与有限展开

（1）获得 $f\left(x\right)=\arcsin x^2$ 关于零点的带Peano余项的Taylor展开

（2）设有 $\exists f^{(n+p)}\left(x_0\right)\in\mathbb{R}$，且 $\exists f^{(n+j)}\left(x_0\right)=0$， $j=1,\cdots,p-1$； $f^{(n+p)}\left(x_0\right)\neq 0$. 研究：

$f\left(x\right)=f\left(x_0\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k+\frac{f^{(n)}\left(x_0+\theta\left(x\right)\cdot\left(x-x_0\right)\right)}{n!}\left(x-x_0\right)^n$ 中 $\theta\left(x\right)$ 的极限

$\lim_{x\to x_0}\theta\left(x\right)$.

(1) 分析：

$$\frac{df}{dx}(x)=\frac{1}{\sqrt{1-x^4}}2x=2x\cdot\left(1-x^4\right)^{-\frac{1}{2}}=2x\left[1+\sum_{k=1}^{n}\binom{-\frac{1}{2}}{k}(-1)^k x^{4k}+o\left(x^{4n+3}\right)\right]$$

$$=2x+2\cdot\sum_{k=1}^{n}\binom{-\frac{1}{2}}{k}(-1)^k x^{4k+1}+o\left(x^{4n+4}\right)$$

有 $f(x)=x^2+\sum_{k=1}^{n}\binom{-\frac{1}{2}}{k}(-1)^k\cdot\frac{x^{4k+2}}{4k+2}+o\left(x^{4n+5}\right)$，式中 $\binom{-\frac{1}{2}}{k}=\frac{(2k-1)!!}{(2k)!!}$

(2) 分析：

$$\frac{f^{(n)}\left(x_0+\theta(x)\cdot\left(x-x_0\right)\right)}{n!}\left(x-x_0\right)^n=\left[f_{(x_0)}^{(n)}+\sum_{j=1}^{p}\frac{f_{(x_0)}^{(n+j)}}{j!}\theta^j(x)\left(x-x_0\right)^j+o\left(\left(x-x_0\right)^p\right)\right]\frac{\left(x-x_0\right)^n}{n!}$$

$$f\left(x\right)=f\left(x_0\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k+\frac{f_{(x_0)}^{(n)}}{n!}\left(x-x_0\right)^n+\sum_{j=1}^{p}\frac{f_{(x_0)}^{(n+j)}}{j!n!}\theta^j(x)\left(x-x_0\right)^{j+n}+o\left(\left(x-x_0\right)^{n+p}\right)$$

有 $f\left(x\right)$ 在 $x_0$ 点无限小展开

$$f\left(x\right)=f\left(x_0\right)+\sum_{k=1}^{n-1}\frac{f^{(k)}\left(x_0\right)}{k!}\left(x-x_0\right)^k+\frac{f_{(x_0)}^{(n)}}{n!}\left(x-x_0\right)^n+\sum_{j=1}^{p}\frac{f^{(n+j)}\left(x_0\right)}{(n+j)!}\left(x-x_0\right)^{j+n}+o\left(\left(x-x_0\right)^{n+p}\right)$$

则有

$$\frac{f^{(n+p)}}{p!n!}\theta^p(x)\left(x-x_0\right)^{n+p}=\frac{f^{(n+p)}\left(x_0\right)}{(n+p)!}\left(x-x_0\right)^{p+n}+o\left(\left(x-x_0\right)^{n+p}\right)$$

有 $\theta^p(x)=\frac{p!n!}{(n+p)!}+o\left(1\right), as\ x\to x_0$，有 $\theta(x)\to\left(\frac{p!n!}{(n+p)!}\right)^{\frac{1}{p}}, as\ x\to x_0$

4. 设有$\exists\lim\limits_{x\to+\infty}\left[f(x)+2\sqrt{x}\cdot\dfrac{df}{dx}(x)\right]=0$，证明：$\exists\lim\limits_{x\to+\infty}f(x)=0$.

分析:

$$f(x)+2\sqrt{x}\frac{df}{dx}(x)=2\sqrt{x}\left[\frac{df}{dx}(x)+\frac{1}{2\sqrt{x}}\cdot f(x)\right]$$

$$=\left.2\sqrt{x}\cdot\frac{d}{dx}\left(e^{\int\frac{1}{2\sqrt{x}}dx}\cdot f(x)\right)\right/e^{\int\frac{1}{2\sqrt{x}}dx}=\frac{\frac{d}{dx}\left(e^{\sqrt{x}}f(x)\right)}{e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}=\frac{\frac{d}{dx}\left(e^{\sqrt{x}}f(x)\right)}{\frac{d}{dx}e^{\sqrt{x}}}$$

$$\sim\frac{e^{\sqrt{x}}f(x)}{e^{\sqrt{x}}}=f(x)$$

5. 设有$\exists\lim\limits_{n\to+\infty}x_n=0$，考虑$y_n=\sum\limits_{k=1}^{n}t_{nk}x_k$，设有$\begin{cases}\sum\limits_{k=1}^{n}|t_{nk}|\le M\in\mathbb{R}^+,\ \forall n\in\mathbb{N}\\ \lim\limits_{n\to+\infty}t_{nk}=0,\ \forall k\in\mathbb{N}\end{cases}$，证明：$\exists\lim\limits_{n\to+\infty}y_n=0$.

分析:

$$y_n=\sum_{k=1}^{n}t_{nk}\cdot x_k=\begin{cases}t_{n1}\cdot x_1+\cdots+t_{nN}\cdot x_N & (RHS1)\\ +t_{nN+1}x_{N+1}+\cdots+t_{nn}x_n & (RHS2)\end{cases},\ |x_n|<\varepsilon,\ \forall n>N_\varepsilon$$

$Step1$:

$$|RHS2|<|t_{nN_\varepsilon+1}||x_{N_\varepsilon+1}|+\ldots+|t_{nn}||x_n|<\varepsilon\sum_{k=N_\varepsilon+1}^{n}|t_{nk}|\le M\varepsilon$$

$Step2$:

$$|RHS1|\le|t_{n1}|\cdot|x_1|+\cdots+|t_{nN}|\cdot|x_{N_\varepsilon}|<\varepsilon,\ \forall n>\max\left\{\tilde{N}_{\varepsilon1},\cdots,\tilde{N}_{\varepsilon,N_\varepsilon},\right\}$$

综上, 有

$$|RHS1|+|RHS2|<(M+1)\varepsilon$$

作平移: $x_n=a+\delta_n,\delta_n\to0$, 有

$$y_n=\sum_{k=1}^{n}t_k\cdot x_k=\sum_{k=1}^{n}t_{nk}(a+\delta_k)=a\cdot\sum_{k=1}^{n}t_{nk}+\sum_{k=1}^{n}t_{nk}\cdot\delta_k$$

其中 $\sum\limits_{k=1}^{n}t_{nk}\to1,\ \sum\limits_{k=1}^{n}t_{nk}\cdot\delta_k\to0,\ as\ n\to+\infty$

（装订线内不要答题）

## 6. 不等式

（1）证明：$\frac{1}{\frac{\alpha_1}{x_1}+\cdots+\frac{\alpha_n}{x_n}} \le x_1^{\alpha_1}\cdots x_n^{\alpha_n} \le \alpha_1 x_1+\cdots+\alpha_n x_n$，$\{x_k\}_{k=1}^{n}\subset\mathbb{R}^+$，$\{\alpha_k\}_{k=1}^{n}\subset\mathbb{R}^+$，$\alpha_1+\cdots+\alpha_n=1$． （2）证明：$\frac{2}{\pi}x\le\sin x\le 1$，$x\in\left[0,\frac{\pi}{2}\right]$

$(1)$ 分析.

$(a)$ 根据Jensen不等式有 $\alpha_1\ln x_1+\cdots+\alpha_n\ln x_n\leqslant\ln\left(\alpha_1 x_1+\cdots\alpha_n x_n\right)$

考虑 $\varphi(x)=\ln x,\forall\in\mathbb{R}^+,\varphi''(x)=-\frac{1}{x^2}$, $\mathbb{R}^+$上上凸

$(b)$ $\ln\frac{1}{\frac{\alpha_1}{x_1}+\cdots+\frac{\alpha_n}{x_n}}\le\alpha_1\ln x_1+\cdots+\alpha_n\ln x_n$

$-\ln\left(\frac{\alpha_1}{x_1}+\cdots+\frac{\alpha_n}{x_n}\right)\le\alpha_1\ln x_1+\cdots+\alpha_n\ln x_n$

有 $\ln\left(\alpha_1\frac{1}{x_1}+\cdots+\alpha_n\frac{1}{x_n}\right)\ge\alpha_1\ln\frac{1}{x_1}+\cdots+\alpha_n\ln\frac{1}{x_n}$

$(2)$ 分析：

$(a)$ 证 $\sin x\le x$

$\delta(x)=x-\sin x,s.t.\delta(0)=0$

$\delta'(x)=1-\cos x>0,\forall x\in\left(0,\frac{\pi}{2}\right)$,有$\delta(x)$在$\left[0,\frac{\pi}{2}\right]$上单调上升

$(b)$ 证 $\frac{2}{\pi}x\le\sin x$

处理一：利用$\varphi(x)=\sin x$的凹凸性

$\varphi'(x)=\cos x,\varphi''(x)=-\sin x\le 0,\forall x\in\left[0,\frac{\pi}{2}\right]$

处理二：$\delta(x)=\sin x-\frac{2}{\pi}x,s.t.\begin{cases}\delta(0)=0\\ \delta\left(\frac{\pi}{2}\right)=0\end{cases}$

有$\delta'(x)=\cos x-\frac{2}{\pi}$,有$x_*=\arccos\frac{2}{\pi},\delta''(x)=-\sin x\le 0,\forall x\in\left[0,\frac{\pi}{2}\right]$

第三部分 分析与证明

（1）① 考虑 $f(x)=\varphi(x)\sin^{\mu}x^{\lambda}$，$\mu\in\mathbb{R}^{+}$，$\lambda\geq 1$ 式中 $\varphi(x)\in C[0,+\infty)$ 且 $\exists\varphi(+\infty)=+\infty$，研究其在 $[0,+\infty)$ 上的一致连续性. ② 研究 $f(x)=x^{\frac{1}{3}}\sin x^{\frac{1}{2}}$ 在 $[0,+\infty)$ 上的一致连续性与极限 $\lim\limits_{x\to+\infty}x^{\frac{1}{3}}\sin x^{\frac{1}{2}}$ 的存在性.

(1) 分析：$x_n=(2n\pi+\delta_n)^{\frac{1}{\lambda}}$，$\delta_n\to 0$

$$f(x_n)=\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)\cdot\sin^{\mu}(2n\pi+\delta_n)=\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)\sin^{\mu}\cdot\delta_n$$

$$=\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)\delta_n^{\mu}\cdot(1+o(1)).$$

考虑 $\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)\cdot x^{\mu}=C\in\mathbb{R}^{+},\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)=\dfrac{C}{x^{\mu}}$

$$\delta_n^{\mu}=\frac{C}{\varphi\left((2n\pi+\delta_n)^{\frac{1}{\lambda}}\right)}\to 0,\ as\ n\to+\infty$$

$$\begin{cases}\tilde{x}_n=(2n\pi+\tilde{\delta}_n)^{\frac{1}{\lambda}},\tilde{\delta}_n\to 0+0, s.t.f(\tilde{x}_n)\to\tilde{C},\text{as } n\to+\infty\\ \hat{x}_n=(2n\pi+\hat{\delta}_n)^{\frac{1}{\lambda}}\cdot\hat{\delta}_n\to 0+0, s.t.f(\hat{x}_n)\to\hat{C},\text{ as } n\to+\infty\end{cases}$$

则有 $\tilde{x}_n-\hat{x}_n\to 0,\text{ as } n\to+\infty$，而 $f(\tilde{x}_n)-f(\hat{x}_n)\to\tilde{C}-\hat{C}\neq 0$

$$\hat{x}_n-\tilde{x}_n=(2n\pi+\hat{\delta}_n)^{\frac{1}{\lambda}}-(2n\pi+\tilde{\delta}_n)^{\frac{1}{\lambda}}=(2n\pi)^{\frac{1}{\lambda}}\left(1+\frac{\hat{\delta}_n}{2n\pi}\right)^{\frac{1}{\lambda}}-(2n\pi)^{\frac{1}{\lambda}}\left(1+\frac{\tilde{\delta}_n}{2n\pi}\right)^{\frac{1}{\lambda}}$$

其中 $\left(1+\dfrac{\hat{\delta}_n}{2n\pi}\right)^{\frac{1}{\lambda}}=1+\dfrac{\hat{\delta}_n}{2n\pi}\cdot\dfrac{1}{\lambda}+\dfrac{\hat{\delta}_n}{2n\pi}o(1),\left(1+\dfrac{\tilde{\delta}_n}{2n\pi}\right)^{\frac{1}{\lambda}}=1+\dfrac{\tilde{\delta}_n}{2n\pi}\cdot\dfrac{1}{\lambda}+\dfrac{\tilde{\delta}_n}{2n\pi}o(1)$

$$\Rightarrow\ \hat{x}_n-\tilde{x}_n=\frac{1}{\lambda}\hat{\delta}_n\frac{1}{(2n\pi)^{1-\frac{1}{\lambda}}}(1+o(1))-\frac{1}{\lambda}\tilde{\delta}_n\frac{1}{(2n\pi)^{1-\frac{1}{\lambda}}}(1+o(1))$$

设有 $1-\dfrac{1}{\lambda}\geq 0$, 即 $\lambda\geq 1$，则有 $\hat{x}_n-\tilde{x}_n\to 0$，$as\ n\to+\infty$

即有 $\varphi(x)\sin^{\mu}x^{\lambda}$, $\begin{cases}\forall\mu\in\mathbb{R}^{+}\\ \forall\lambda\geq 1\end{cases}$ 非一致连续函数.

(2) 分析：$f(x)$对应(1)里$\begin{cases}\mu=1\\ \lambda=\dfrac{1}{2}\end{cases}$的情况

取$\begin{cases}\tilde{x}_n\to+\infty, f(\tilde{x}_n)\to\tilde{C}\\ \hat{x}_n\to+\infty, f(\hat{x}_n)\to\hat{C}\end{cases}$，破坏l 序列性刻画，所以极限显然不存在

$f'(x)=\dfrac{1}{3}x^{-\frac{2}{3}}\sin x^{\frac{1}{2}}+x^{\frac{1}{3}}\cos x^{\frac{1}{2}}\dfrac{1}{2}x^{-\frac{1}{2}}$，导数可控$(x\gg1)$，故一致连续.

（2）设$T$为正常数，若有$\psi(x)$、$\varphi(x)$在$[a,+\infty)$上满足

$$\begin{cases}(1)\quad \varphi(x+T)>\varphi(x),\ x\in[a,+\infty);\\ (2)\quad \lim\limits_{x\to+\infty}\varphi(x)=+\infty,\ \psi(x),\varphi(x)\text{在}[a,+\infty)\text{的每个有界子区间上有界;}\\ (3)\quad \exists\lim\limits_{x\to+\infty}\dfrac{\psi(x+T)-\psi(x)}{\varphi(x+T)-\varphi(x)}=l\in\mathbb{R},\end{cases}$$

，证明：$\exists\lim\limits_{x\to+\infty}\dfrac{\psi(x)}{\varphi(x)}=l$.

分析：

$\exists\lim\limits_{x\to+\infty}\dfrac{\psi(x+T)-\psi(x)}{\varphi(x+T)-\varphi(x)}=l\in\mathbb{R}\ \Rightarrow\ \dfrac{\psi(x+T)-\psi(x)}{\varphi(x+T)-\varphi(x)}=l+o(1),\ as\ x\to+\infty$

$\psi(x+T)-\psi(x)=l\cdot[\varphi(x+T)-\varphi(x)]+o(1)\cdot[\varphi(x+T)-\varphi(x)],\ as\ x\to+\infty\ \Rightarrow$

$$\begin{cases}\psi(x+T)-\psi(x)=l\cdot[\varphi(x+T)-\varphi(x)]+o(1)\cdot[\varphi(x+T)-\varphi(x)]\\ \psi(x+2T)-\psi(x+T)=l\cdot[\varphi(x+2T)-\varphi(x+T)]+o(1)\cdot[\varphi(x+2T)-\varphi(x+T)]\\ \psi(x+3T)-\psi(x+2T)=l\cdot[\varphi(x+3T)-\varphi(x+2T)]+o(1)\cdot[\varphi(x+3T)-\varphi(x+2T)]\\ \cdots\\ \psi(x+nT)-\psi(x+(n-1)\cdot T)=l\cdot[\varphi(x+nT)-\varphi(x+(n-1)\cdot T)]+o(1)\cdot[\varphi(x+nT)-\varphi(x+(n-1)\cdot T)]\end{cases}\Rightarrow$$

$\psi(x+nT)-\psi(x)=l\cdot[\varphi(x+nT)-\varphi(x)]+\sum\limits_{k=1}^{n}o(1)\cdot[\varphi(x+kT)-\varphi(x+(k-1)\cdot T)],\ as\ x\to+\infty$

$$\frac{\psi(x+nT)}{\varphi(x+nT)}-l=\frac{\psi(x)}{\varphi(x+nT)}-l\cdot\frac{\varphi(x)}{\varphi(x+nT)}+\frac{\sum\limits_{k=1}^{n}o(1)\cdot[\varphi(x+kT)-\varphi(x+(k-1)\cdot T)]}{\varphi(x+nT)}$$

$$\left|RHS_3\right| \le \left|\frac{\sum_{k=1}^{n} o(1)\cdot\left[\varphi(x+kT)-\varphi(x+(k-1)\cdot T)\right]}{\varphi(x+nT)}\right| \le \frac{\sum_{k=1}^{n}\left|o(1)\right|\cdot\left|\varphi(x+kT)-\varphi(x+(k-1)\cdot T)\right|}{\varphi(x+nT)}$$

$$< \varepsilon\cdot\frac{\sum_{k=1}^{n}\left|\varphi(x+kT)-\varphi(x+(k-1)\cdot T)\right|}{\varphi(x+nT)} = \varepsilon\cdot\frac{\left|\varphi(x+nT)-\varphi(x)\right|}{\varphi(x+nT)} \le \varepsilon\cdot\left(1+\frac{\varphi(x)}{\varphi(x+nT)}\right),\ \forall x \ge \delta_\varepsilon$$

$$\left|RHS_2\right| \le |l|\cdot\frac{\varphi(x)}{\varphi(x+nT)} \le |l|\cdot\frac{M_{\varphi,\varepsilon}}{\varphi(x+nT)},\ \forall x\in\left[\delta_\varepsilon,\delta_\varepsilon+T\right]$$

$$\le |l|\cdot\varepsilon,\ \forall x\in\left[\delta_\varepsilon,\delta_\varepsilon+T\right],\ \forall n > N_{\varphi,\frac{M_{\varphi,\varepsilon}}{\varepsilon}}$$

$$\left|RHS_1\right| \le \left|\frac{\psi(x)}{\varphi(x+nT)}\right| \le \frac{M_{\psi,\varepsilon}}{\varphi(x+nT)},\ \forall x\in\left[\delta_\varepsilon,\delta_\varepsilon+T\right]$$

$$\le \varepsilon,\ \forall x\in\left[\delta_\varepsilon,\delta_\varepsilon+T\right],\ \forall n > N_{\psi,\frac{M_{\psi,\varepsilon}}{\varepsilon}}$$

$$\Rightarrow \left|\frac{\psi(x+nT)}{\varphi(x+nT)}-l\right| < \varepsilon+|l|\cdot\varepsilon+\varepsilon\cdot(1+\varepsilon),\ \forall x\in\left[\delta_\varepsilon,\delta_\varepsilon+T\right],\ \forall n > \max\left\{N_{\varphi,\frac{M_{\varphi,\varepsilon}}{\varepsilon}}, N_{\psi,\frac{M_{\psi,\varepsilon}}{\varepsilon}}\right\}$$

$$\Rightarrow \left|\frac{\psi(y)}{\varphi(y)}-l\right| < \varepsilon+|l|\cdot\varepsilon+\varepsilon\cdot(1+\varepsilon),\ \forall y > \delta_\varepsilon+\max\left\{N_{\varphi,\frac{M_{\varphi,\varepsilon}}{\varepsilon}}, N_{\psi,\frac{M_{\psi,\varepsilon}}{\varepsilon}}\right\}\cdot T$$

$$\Rightarrow \exists \lim_{x\to+\infty}\frac{\psi(x)}{\varphi(x)} = l$$

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