QP $\min \frac{1}{2}x^TPx + q^Tx$

s.t. $-1 \le x \le 1$

C

$\nabla f(x^*)^T(y-x^*)$

$\forall y \in C$

1. (15 points) Consider the problem

$$\min_x \frac{\|Ax-b\|_1^2}{1-\|x\|_\infty} \quad \text{s.t.} \|x\|_\infty \le 1/2.$$

Derive a SOCP reformulation for the problem, i.e., the objective is affine, and all the constraints are either linear or second order cone (composite with affine functions).

2. (15 points) Let $f_1, \dots, f_m$ be convex functions on $\mathbb{R}^n$. Define

$$g(x) = \inf_{x_1,\dots,x_m} \{f_1(x_1) + f_2(x_2) + \dots + f_m(x_m) \mid x_1 + x_2 + \dots + x_m = x\}.$$

1) (5 points) Show that $g(x)$ is convex.

2) (10 points) Prove that the conjugate function $g^*(y) = f_1^*(y) + f_2^*(y) + \cdots + f_m^*(y)$.

3. (25 points) Consider the problem of minimizing the point-wise maximum of a finite collection of functions:

$$\min_x \max_{1\le i\le m} f_i(x), \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ are smooth convex functions. This problem is equivalent to the following form

$$\min_x \max_{y\in Y} \sum_{i=1}^m y_i f_i(x), \tag{2}$$

where $Y = \left\{(y_1, \cdots, y_m)^T \mid \sum_{i=1}^m y_i = 1, y_i \ge 0\right\}$ is a probability simplex.

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3. (25 points) Consider the problem of minimizing the point-wise maximum of a finite collection of functions:

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1) (8 points) Derive the optimality conditions of problem (2). Denote by ν the Lagrange multiplier of $y \ge 0$ and by μ the Lagrange multiplier of $\sum_{i=1}^m y_i = 1$. **Hint:** You may first derive the KKT conditions for the inner maximization problem (fixing x) and then obtain the first-order optimality conditions for the outer minimization problem (fixing y). Note that the inner problem is a maximization problem.

2) (7 points) For $y \in Y$, define the active set $\mathcal{A}[y] = \{i \in [m] \mid y_i = 0\}$ and the inactive set $\mathcal{I}[y] = \{i \in [m] \mid y_i > 0\}$. Define the top coordinate set $\mathcal{T}(x)$ as the collection of all indexes associated with the largest value of $[f_1(x), \dots, f_m(x)]^\top$, i.e., $\mathcal{T}(x) = \{i : f_i(x) = \max_{j=1,\dots,m} f_j(x)\}$. For (x^*, y^*) satisfying the KKT conditions, prove that $\mathcal{I}[y^*] \subseteq \mathcal{T}(x^*)$. In addition, if the strict complementarity assumption holds (i.e., suppose ν_i is the Lagrange multiplier of $y_i \ge 0$, if $y_i = 0$ then $\nu_i > 0$), prove that $\mathcal{I}[y^*] = \mathcal{T}(x^*)$.

3) (5 points) Assume the strict complementarity assumption holds. Define the matrix

$$M(x) = \begin{bmatrix} \nabla f_{j_1}(x), & \dots, \nabla f_{j_{|\mathcal{T}(x)|}}(x) \\ & \mathbf{1}_{|\mathcal{T}(x)|}^\top \end{bmatrix},$$

where $\mathcal{T}(x) = \{j_1, \cdots, j_{|\mathcal{T}(x)|}\}$. Prove that for (x^*, y^*) satisfying the KKT conditions, the matrix $M(x^*)$ is of full column rank. **Hint:** You may assume that there exists a nonzero vector v such that $M(x^*)v = 0$, and use v and y^* to construct a counterexample.

4) (5 points) Under the assumption of 3), prove that for any solution x^* of (1), there exists only one $y \in Y$ ... solution of (2), and there exists only one (μ, ν) such that (x^*, y, μ, ν) satisfies the

such that (x^*, y) is the solution of (2), and there exists only one (μ, ν) such that (x^*, y, μ, ν)
KKT conditions.