（装订线内不要答题）

复旦大学大数据学院

2020年春季学期课程期末考试卷答题纸（样卷）

☒ **A 卷**　　☐**B 卷**

课程名称：最优化方法
课程代码：DATA130026.01
开课院系：大数据学院 考试形式：闭卷

姓　名：__________ **学　号：**_________ **专　业：**______________

声明：我已知悉学校对于考试纪律的严肃规定，将秉持诚实守信宗旨，严守考试纪律，不作弊，不剽窃；若有违反学校考试纪律的行为，自愿接受学校严肃处理。
学生（签名）：_____________
年　　月　　日

题　目	1	2	3	4	5	6	总　分
得　分							

1. (10 points) Please answer true or false. No explanation is needed. A correct answer is worth 2 points, no answer 0 points, a wrong answer -1.

   (a) The feasible set of a linear program (LP) in standard form is always bounded.

   (b) Consider an LP. If the dual problem is infeasible, then so is the primal problem.

   (c) The problem $\max \sum_{j=1}^{n} c_j|x_j|$ subject to $\sum_{j=1}^{n} a_j|x_j| \le b$, with $c_j, a_j \ge 0$, can be modeled as a linear optimization problem.

   (d) Given a local optimum $\bar{x}$ for a nonlinear optimization problem, it always satisfies the KKT conditions when the gradients of the active constraints and the gradients of the equality constraints at the point $\bar{x}$ are linearly independent.

   (e) For a quadratic function $f(x) = x^T Ax + b^T x + c$ with $A \succ 0$, the convergence rate of Newton's method depends on the condition number of the matrix A.

2. (10 points) A function f is called log-convex if $f(x) > 0$, for all $x \in \mathrm{dom}(f)$ and $\log(f(x))$ is convex.

   (a) (5 points) If f is log-convex, then f is also convex.

   (b) (5 points) f is log-convex if and only if $\forall \lambda \in [0,1], \forall x, y \in \mathrm{dom}(f)$, we have $f(\lambda x + (1-\lambda)y) \le f(x)^{\lambda} f(y)^{1-\lambda}$.

3. (10 points) Write down the conjugate function of $f(z) = \log(1 + e^{-z})$. (Recall that the conjugate function $f^*(y) := \sup_z yz - f(z)$.)

4. (30 points) Let $a \in \mathbb{R}^n_+$ with $a \neq 0$ and $b > 0$ be given. Consider the following problem:

$$\begin{aligned} \text{(P)} \quad \min \quad & \frac{1}{2}\|x\|_2^2 \\ \text{s.t.} \quad & a^T x = b, \\ & x \ge 0. \end{aligned}$$

   (a) (5 points) Show that the above problem is feasible, i.e., there exists at least one feasible solution.

   (b) (10 points) Show that an optimal solution always exists, and that it is unique.

   (c) (5 points) Write down the KKT conditions associated with problem (P).

   (d) (10 points) Suppose that you know that the above KKT conditions are necessary and sufficient for optimality. Use the KKT conditions, or otherwise, express the optimal solution to problem (P) in terms of a and b.

5. (20 points) Consider the Quadratically Constrained Quadratic Program (QCQP):

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T Q_0 x + 2b_0^T x \\ \text{s.t.} \quad & x^T Q_i x + 2b_i^T x + c_i \le 0, \ i = 1, \dots, m \\ & Ax = b, \end{aligned}$$

where $Q_i \in \mathbb{S}^n$ (i.e., $n \times n$ symmetric matrix), $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$, $i = 0, \dots, m$, $A \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^p$ are given.

   (a) (5 points) What constraints must each Q_i satisfy for the problem to be convex?

   (b) (10 points) Derive the dual of the problem.

   (c) (5 points) Derive the SDP relaxation of the problem.

6. (20 points) Let $L : \mathbb{R}^l \to \mathbb{R}$ be a convex function, $A \in \mathbb{R}^{n \times l}$ and $v \in \mathbb{R}^l$. Denote $L^* : \mathbb{R}^l \to \mathbb{R}$ the conjugate function of L. Show that the dual problem of

$$\min_{w \in \mathbb{R}^n, b \in \mathbb{R}} L(A^T w - bv) + \frac{1}{2}\|w\|_2^2,$$

can be written as

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^l} \quad & L^*(-\alpha) + \frac{1}{2}\|A\alpha\|_2^2 \\ \text{s.t.} \quad & v^T \alpha = 0. \end{aligned}$$